Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications 116 (2006) 1511–1529

stochastic processes and their applications

www.elsevier.com/locate/spa

Annealed asymptotics for the parabolic Anderson model with a moving catalyst

Jürgen Gärtner[a], Markus Heydenreich[b,*]

[a] *Institut für Mathematik, Technische Universität Berlin, MA 7-5, Str. des 17. Juni 136, 10623 Berlin, Germany*
[b] *Department of Mathematics and Computer Science, Technische Universiteit Eindhoven, P.O. Box 513, 5600 MB Eindhoven, The Netherlands*

Received 3 June 2005; received in revised form 4 April 2006; accepted 4 April 2006
Available online 27 April 2006

Abstract

This paper deals with the solution u to the parabolic Anderson equation $\partial u/\partial t = \kappa \Delta u + \xi u$ on the lattice $\mathbb{Z}^d$. We consider the case where the potential ξ is time-dependent and has the form $\xi(t, x) = \delta_0(x - Y_t)$ with Y_t being a simple random walk with jump rate $2d\varrho$. The solution u may be interpreted as the concentration of a *reactant* under the influence of a single *catalyst* particle Y_t.

In the first part of the paper we show that the moment Lyapunov exponents coincide with the upper boundary of the spectrum of certain Hamiltonians. In the second part we study intermittency in terms of the moment Lyapunov exponents as a function of the model parameters κ and ϱ.

MSC: primary 60H25; 82C44; secondary 60F10; 35B40

Keywords: Parabolic Anderson problem; Catalytic random medium; Intermittency; Moment Lyapunov exponents

1. Introduction

1.1. The parabolic Anderson problem and its interpretation

The main object of our investigation is the solution $u: \mathbb{R}^+ \times \mathbb{Z}^d \to \mathbb{R}^+$ to the Cauchy problem for the heat equation with a random time-dependent potential:

* Corresponding author. Tel.: +31 40 2474332; fax: +31 40 2465995.
E-mail addresses: jg@math.tu-berlin.de (J. Gärtner), m.o.heydenreich@tue.nl (M. Heydenreich).

0304-4149/$ - see front matter
doi:10.1016/j.spa.2006.04.002

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \kappa \Delta u(t,x) + \xi(t,x)\, u(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{Z}^d, \\ u(0,x) = 1, & x \in \mathbb{Z}^d. \end{cases} \tag{1.1}$$

Here, $\kappa \in \mathbb{R}^+$ is a diffusion constant and Δ is the discrete Laplacian acting on $f\colon \mathbb{Z}^d \to \mathbb{R}$ as

$$\Delta f(x) = \sum_{y \sim x} [f(y) - f(x)],$$

while

$$\xi(t) = \left\{\xi(t,x) \mid x \in \mathbb{Z}^d\right\}, \quad t \in \mathbb{R}^+,$$

is an $\mathbb{R}$-valued random field evolving over time that "drives" the equation. Problem (1.1) is referred to as the *parabolic Anderson model*. It is the parabolic analogue of the Schrödinger equation with a time-dependent random potential.

A popular heuristic interpretation of the model arises from population dynamics. In this context the function $u(t,x)$ describes the mean number of particles present at x at time t when starting with one particle per site. Particles perform independent random walks on $\mathbb{Z}^d$ with jump rate $2d\kappa$ and split into two at rate ξ if $\xi > 0$ (source) or die at rate $-\xi$ if $\xi < 0$ (sink).

If ξ is a nonnegative field, then we can interpret the problem in (1.1) also as a linearized model of chemical reactions. In this case, the solution of the equation describes the evolution of reactant particles under the influence of a catalyst medium ξ. More precisely, u describes the expected number of reactant particles if its time evolution is governed by the following rules:

(i) at time $t = 0$, each lattice site is occupied by one reactant;
(ii) reactants act independently of each other;
(iii) a reactant at x jumps to a neighboring site y at rate κ;
(iv) a reactant at x splits into two at rate $\xi(t,x)$.

Another example is mathematical modeling in evolution theory. Considering a fixed size population, one may describe its evolution by the Fisher–Eigen equation of population genetics which is a version of (1.1). Hereby $\mathbb{Z}^d$ represents the space of phenotypes, Δ describes mutation and ξ is the fitness. See e.g. Ebeling et al. [5, Sect. 2] for such an approach.

Characteristically for the parabolic Anderson model, the two terms on the right hand side of Eq. (1.1) compete with each other. The diffusion induced by Δ tends to make u flat whereas ξ tends to make u bumpy. In the context of population dynamics, there is a competition between individuals spreading out by diffusion and clumping around sources.

Studying problem (1.1), we distinguish between the *quenched* setting which describes the almost sure behaviour of u conditioned on ξ, and the *annealed* setting, where we average over ξ. The present paper deals with the annealed setting.

The theory currently available for the model covers various forms of the potential ξ. In the present paper we consider the case where ξ has the form

$$\xi(t,x) = \delta_{Y_t}(x), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{Z}^d, \tag{1.2}$$

where $(Y_t)_{t \geq 0}$ is a random walk with generator $\varrho\Delta$ starting at the origin and $\delta_y(x)$ is the Kronecker symbol. The corresponding expectation will be denoted by $\langle \cdot \rangle$. The parameter $\varrho \in [0,\infty)$ is the diffusion constant of the catalyst. In the context of chemical reactions, we can interpret ξ as the reaction rate induced by a single catalyst particle, which performs a random walk in $\mathbb{Z}^d$ with jump rate $2d\varrho$. Reactants split into two at rate 1 if they are at the same lattice site as the catalyst. Gärtner and den Hollander [6] have been investigating this kind of problem with infinitely many independently moving catalysts starting from a homogeneous Poisson field. We describe their results in Section 1.4.

For a general discussion of the parabolic Anderson model, the reader is referred to the survey by Gärtner and König [7].

Our main tool for the analysis of the solution to the parabolic Anderson problem is the *Feynman–Kac formula*. It states that a solution to the differential equation (1.1) with a bounded initial datum u_0 is given by

$$u(t,x) = \mathbb{E}_x^X \exp\left\{\int_0^t \xi(t-s, X_s)\,\mathrm{d}s\right\} u_0(X_t), \tag{1.3}$$

where $(X_s)_{s\geq 0}$ is a random walk on $\mathbb{Z}^d$ with generator $\kappa\Delta$ and expectation $\mathbb{E}_x^X$ when starting at x.

1.2. Lyapunov exponents and intermittency

The aim of the present paper is to study the *p-th moment Lyapunov exponent*

$$\lambda_p = \lambda_p(\kappa, \varrho) = \lim_{t\to\infty} \frac{1}{t} \log\langle u(t,x)^p\rangle \tag{1.4}$$

for $p \in \mathbb{N}$ as a function of the model parameters $\kappa, \varrho \in [0,\infty)$.

We will see in Theorem 1.2 below that the finite limit (1.4) exists for all $p \in \mathbb{N}$ and is independent of x.

Definition 1.1 (*Intermittency*). For $p \in \mathbb{N}\setminus\{1\}$, we call the parabolic Anderson problem (1.1) *p-intermittent*, if the Lyapunov exponents satisfy the *strict* inequality

$$\frac{\lambda_{p-1}}{p-1} < \frac{\lambda_p}{p}. \tag{1.5}$$

We say the system is *fully intermittent*, if the system is p-intermittent for all $p \in \mathbb{N}\setminus\{1\}$.

Note that, by Hölder's inequality, we always have that $\lambda_{p-1}/(p-1) \leq \lambda_p/p$.

So far there exists no fully satisfactory rigorous mathematical definition of intermittency. The above definition goes back to physicists (see e.g. [10]) and is very much in the spirit of [8] and [1]. Generally, intermittency corresponds to a very irregular behaviour of the solution u. In the case of a nonnegative ergodic random field ξ, the solution u is also ergodic and exhibits very high, but more and more widely spaced *peaks* absorbing its total mass. See [8, Sect. 1.1] or [7, Sect. 1.3] for details.

To illustrate intermittency, we assume that u is p-intermittent and choose a level α such that $\lambda_{p-1}/(p-1) < \alpha < \lambda_p/p$. As in the above references, a simple application of Chebyshev's inequality shows that, on the one hand, $\langle \mathbb{1}_{\{u(t,0)>\mathrm{e}^{\alpha t}\}}\rangle \to 0$ exponentially fast as $t\to\infty$ and, on the other hand, $\langle u(t,0)^p\rangle \sim \langle u(t,0)^p\, \mathbb{1}_{\{u(t,0)>\mathrm{e}^{\alpha t}\}}\rangle$. Hence, asymptotically the p-th moment is 'generated' by the exponentially rare event that the solution exceeds the high level $\mathrm{e}^{\alpha t}$. In our context, for lack of homogeneity, there is no direct geometric interpretation of this fact via Birkhoff's ergodic theorem.

For our model, we will see that p-intermittency implies q-intermittency for all $q > p$. We will find qualitatively different intermittency behaviours for dimension $d = 1, 2$ on the one hand and $d \geq 3$ on the other hand.

1.3. Results

From now on we stick to the parabolic Anderson problem (1.1) with the single catalyst potential (1.2). Our first result establishes the existence of the limit (1.4) and provides a spectral characterization of the Lyapunov exponents.

Given $p \in \mathbb{N}$, let B^p denote the operator in $\ell^2(\mathbb{Z}^{pd})$ given by

$$B^p f(x_1, \ldots, x_p) = \sum_{\substack{e \in \mathbb{Z}^d \\ |e|=1}} [f(x_1 + e, \ldots, x_p + e) - f(x_1, \ldots, x_p)],$$

$$f \in \ell^2(\mathbb{Z}^{pd}), x_1, \ldots, x_p \in \mathbb{Z}^d, \tag{1.6}$$

and introduce the Hamilton operator

$$\mathcal{H}^p := \kappa \Delta_1 + \cdots + \kappa \Delta_p + \varrho B^p + \delta_0^{(1)} + \cdots + \delta_0^{(p)} \tag{1.7}$$

on $\ell^2(\mathbb{Z}^{pd})$. Here Δ_i is the discrete Laplacian acting on the i-th argument and $\delta_0^{(i)}(x_1, \ldots, x_p) = 1$ if $x_i = 0$ and 0 otherwise ($i = 1, \ldots, p$). Note that $B^1 = \Delta$.

The following theorem links the asymptotic behaviour of $\langle u(t,x)^p \rangle$ as $t \to \infty$ to the ℓ^2-spectrum $\mathrm{Sp}(\mathcal{H}^p)$ of the operator $\mathcal{H}^p$.

Theorem 1.2 (*Existence and Spectral Characterization*). *Let $\kappa, \varrho \geq 0$, $\kappa + \varrho > 0$. For each $p \in \mathbb{N}$, the Lyapunov exponent*

$$\lambda_p = \lim_{t \to \infty} \frac{1}{t} \log \langle u(t,x)^p \rangle$$

exists, is finite and independent of $x \in \mathbb{Z}^d$, and

$$\lambda_p = \sup \mathrm{Sp}(\mathcal{H}^p). \tag{1.8}$$

In the case $\kappa + \varrho = 0$, this is still valid for $x = 0$.

We prove Theorem 1.2 in Section 2.

We are interested in deriving properties of $\lambda_p = \lambda_p(\kappa, \varrho)$ as a function of the parameters κ and ϱ. According to Theorem 1.2, this can be done by analyzing the spectrum $\mathrm{Sp}(\mathcal{H}^p)$. To this end we denote

$$G_d(\mu) := \Big((\mu - \Delta)^{-1} \delta_0, \delta_0 \Big)_{\ell^2(\mathbb{Z}^d)} = \int_0^\infty \mathrm{e}^{-\mu t} p_t(0)\, \mathrm{d}t, \quad \mu \in \mathbb{R}, \tag{1.9}$$

where p_t is the transition function of a random walk with generator Δ. We will further use the abbreviation $G_d := G_d(0)$. Hence, for dimension $d = 1, 2$, $G_d = \infty$, whereas for $d \geq 3$, $G_d < \infty$. Next we introduce the quantity

$$\mu(\kappa) := \sup \mathrm{Sp}(\kappa \Delta + \delta_0). \tag{1.10}$$

It is well known that the ℓ^2-spectrum of $\kappa \Delta + \delta_0$ has the form

$$\mathrm{Sp}(\kappa \Delta + \delta_0) = [-4d\kappa, 0] \cup \{\mu(\kappa)\},$$

where

$$\mu(\kappa) \begin{cases} = 0, & \text{if } \kappa \geq G_d, \\ > 0, & \text{if } \kappa < G_d. \end{cases} \tag{1.11}$$

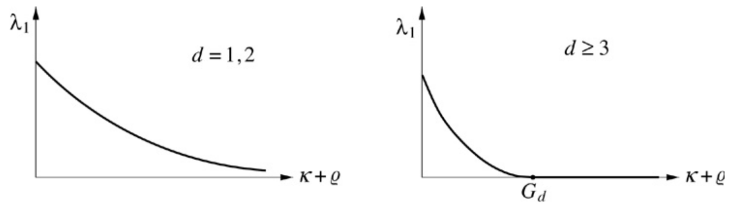

Fig. 1. The qualitative behaviour of λ_1.

In the latter case, $\mu(\kappa)$ is the unique positive solution to $G_d(\mu) = \kappa$. It is the principal eigenvalue of $\kappa\Delta + \delta_0$, which is simple and corresponds to a positive eigenfunction. Furthermore, $\mu(\kappa)$ is convex and nonincreasing in κ (cf. e.g. [6, Lemma 1.3.1]).

The case $p = 1$ can be solved completely, since

$$\mathcal{H}^1 = (\kappa + \varrho)\Delta + \delta_0 \tag{1.12}$$

and hence, by Theorem 1.2,

$$\lambda_1(\kappa, \varrho) = \sup \mathrm{Sp}(\mathcal{H}^1) = \mu(\kappa + \varrho). \tag{1.13}$$

Combining this with (1.11), we obtain the following conclusion. For dimension $d = 1, 2$, the first moment $\langle u(t, x)\rangle$ always grows exponentially fast, whereas for dimension $d \geq 3$ we have exponential growth if $\kappa + \varrho$ falls below the critical value G_d. Otherwise $\langle u(t, x)\rangle$ grows only subexponentially (see Fig. 1).

Remark. The case of an arbitrary strength $\gamma > 0$ of the catalyst, where (1.2) is replaced by

$$\xi(t, x) = \gamma\, \delta_{Y_t}(x), \tag{1.14}$$

can be reduced to $\gamma = 1$ by scaling. To see this, we consider the solution $u_{\kappa,\varrho,\gamma}$ to the parabolic Anderson problem (1.1) with potential (1.14). It follows that $u_{\kappa,\varrho,\gamma}(t, x)$ and $u_{\kappa/\gamma,\varrho/\gamma,1}(\gamma t, x)$ have the same distribution. Consequently, the corresponding Lyapunov exponent $\lambda_p(\kappa, \varrho, \gamma) = \lim_{t\to\infty} t^{-1} \log\langle u_{\kappa,\varrho,\gamma}(t, x)^p\rangle$ satisfies

$$\lambda_p(\kappa, \varrho, \gamma) = \gamma \cdot \lambda_p\left(\frac{\kappa}{\gamma}, \frac{\varrho}{\gamma}, 1\right).$$

Because of this, we set $\lambda_p(\kappa, \varrho) = \lambda_p(\kappa, \varrho, 1)$ and study the qualitative behaviour of the Lyapunov exponents as a function of κ and ϱ only.

We next consider the case $\varrho = 0$ when the catalyst is fixed to its starting position 0. Then the random field ξ is time-independent.

Lemma 1.3 (*The Case* $\varrho = 0$). *For all* $p \in \mathbb{N}$,

$$\frac{\lambda_p(\kappa, 0)}{p} = \mu(\kappa), \quad \kappa \in [0, \infty). \tag{1.15}$$

This result is specifically important for the analysis of λ_p for large p. The statement of the lemma implies that in the setting of a fixed catalyst ($\varrho = 0$) the system is not intermittent for any $p \in \mathbb{N}$. We will see that $\mu(\kappa)$ is an upper bound on $\lambda_p(\kappa, \varrho)/p$.

The case $\kappa = 0$ can be treated similarly. In the language of chemical kinetics, this corresponds to fixed reactants waiting for the catalyst passing by.

Lemma 1.4 (*The Case $\kappa = 0$*). *For all $p \in \mathbb{N}$,*

$$\frac{\lambda_p(0,\varrho)}{p} = \lambda_1(0,\varrho/p) = \mu(\varrho/p), \quad \varrho \in [0,\infty). \tag{1.16}$$

We prove Lemmas 1.3 and 1.4 in Section 3.2.

Using properties of μ, we summarize that in the case $\kappa = 0$, the system is p-intermittent if and only if $0 < \varrho < p\,G_d$. In particular, it is fully intermittent if $0 < \varrho < G_d$.

As a main result for the general behaviour of $\lambda_p(\kappa,\varrho)$ we obtain the following theorem.

Theorem 1.5 (*Properties of λ_p*).

(i) *For each $p \in \mathbb{N}$, the function $\lambda_p(\kappa,\varrho)$, $(\kappa,\varrho) \in [0,\infty)^2$, is continuous, convex, nonincreasing in κ and ϱ, and*

$$\lambda_p(\kappa,\varrho) = 0 \quad \text{for } \kappa \geq G_d. \tag{1.17}$$

(ii) *For all $\kappa, \varrho \in [0,\infty)$,*

$$\frac{\lambda_p(\kappa,\varrho)}{p} \nearrow \mu(\kappa) \quad \text{as } p \nearrow \infty. \tag{1.18}$$

The proof of Theorem 1.5 is given in Section 3.3.

Finally, we state our result on intermittency.

Theorem 1.6 (*Intermittency*). *Let $\varrho > 0$. If $0 \leq \kappa < G_d$, then there exists $p \in \mathbb{N} \setminus \{1\}$ such that the system is p-intermittent, whereas for $\kappa \geq G_d$ the system is not intermittent. Furthermore, for $\kappa + \varrho < G_d$, the system shows full intermittency.*

Except for the statement on full intermittency, this follows from our previous statements, where we used that p-intermittency implies q-intermittency for $q > p$ (cf. Section 3.1). A complete proof of the theorem is given in Section 3.4.

For completeness, we recall from Lemma 1.3 that, for $\varrho = 0$, all curves $\lambda_p(\kappa,0)/p$ coincide with $\mu(\kappa)$ and thus the system is not intermittent. Taking into account that $G_d = \infty$ for dimension $d = 1, 2$, we conclude from Theorem 1.6 that for these dimensions the system shows full intermittency for all $\kappa \in [0,\infty)$, $\varrho \in (0,\infty)$.

1.4. Related work

There exist a wide variety of papers on the parabolic Anderson model with a *time-independent* random field ξ, see the survey by Gärtner and König [7]. The theory for the *time-dependent* parabolic Anderson model is less developed. Let us briefly mention the annealed results obtained in [1,9] and [6].

The monograph by Carmona and Molchanov [1] provides a complete analysis of the moment Lyapunov exponents in the case of a white noise potential

$$\xi(t,x) = \dot{W}_x(t), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{Z}^d, \tag{1.19}$$

with $\{(W_x(t))_{t\geq 0} \mid x \in \mathbb{Z}^d\}$ being a collection of independent Brownian motions and Eq. (1.1) treated in the Itô sense. They show that

$$\lambda_p = \sup \operatorname{Sp}\left(\kappa(\Delta_1 + \cdots + \Delta_p) + V_p\right),$$

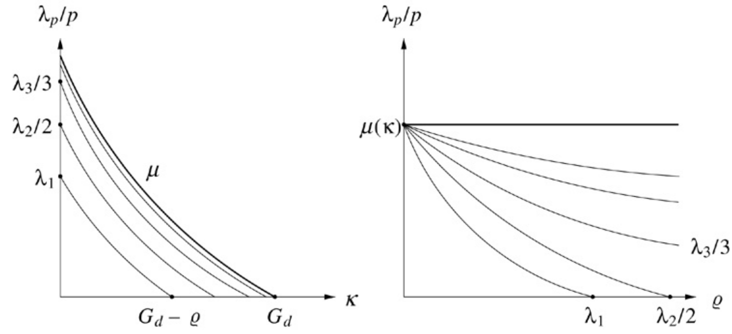

Fig. 2. The asymptotic behaviour of λ_p/p for large p for dimension $d \geq 3$. On the left the variation due to κ for fixed $\varrho > 0$ and on the right the variation due to ϱ for fixed $\kappa \in (0, G_d)$. If $\kappa \geq G_d$, then all curves in the right figure coincide with the horizontal axis.

where

$$V_p(x_1, \ldots, x_p) = \sum_{1 \leq j < k \leq p} \delta_0(x_j - x_k), \quad x_1, \ldots, x_p \in \mathbb{Z}^d.$$

The intermittency behaviour is similar to our model in Fig. 2 due to the similar spectral representation. The essential difference is that λ_p as a function of κ obeys $\lambda_1(\kappa) = 0$, because $V_1 = 0$. Therefore the system is p-intermittent if and only if $\lambda_p > 0$. Furthermore, they obtain a different behaviour for large p: $\lambda_p/p \to \infty$ as $p \to \infty$.

Kesten and Sidoravicius [9] consider a spatially homogeneous system of two types of particles, A (catalyst) and B (reactant), performing independent random walks on the lattice, such that:

(i) B-particles split into two at a rate that is the number of A-particles present at the same lattice site;
(ii) ϱ and κ are the diffusion constants of the A- and B-particles, respectively;
(iii) ν and 1 are the initial intensities of the A- and B-particles, respectively;
(iv) B-particles die at a rate $\delta > 0$.

This corresponds to our model in (1.1) where the potential ξ is given by

$$\xi(t, x) = \sum_k \delta_0(x - Y_k(t)) - \delta, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{Z}^d,$$

with $\{Y_k(t); t \geq 0, k \in \mathbb{N}\}$ being a collection of independent random walks with generator $\varrho\Delta$ starting from a homogeneous Poisson field with intensity $\nu \in \mathbb{R}^+$. Then, $u(t, x)$ is the average number of B-particles at site x at time t conditioned on the evolution of the A-particles. The main focus of Kesten and Sidoravicius is on survival versus extinction. They have shown that for dimension $d = 1, 2$, for any choice of the parameters, the average number of B-particles per site tends to infinity faster than exponential. For dimension $d \geq 3$ with δ sufficiently large, the average number of B-particles per site tends to zero exponentially fast.

The qualitative behaviour of the moments is different from the above in the model considered by Gärtner and den Hollander [6]. They show that there is a strongly catalytic regime where the moments $\langle u(t, 0)^p \rangle$ grow superexponentially fast. This is always the case for dimension $d = 1, 2$, and also for dimension $d \geq 3$ for $\varrho/p < G_d$ (independent of κ). Otherwise, the finite moment Lyapunov exponents (1.4) exist. It is shown that their intermittency behaviour as a function of κ

is different for $d = 3$ and for $d \geq 4$. For $d = 3$, the moment Lyapunov exponents are expressed via the polaron variational problem.

Our model (1.1) itself is similar to that by Gärtner and den Hollander, but our methods and results are more closely related to those of Carmona and Molchanov. Their analysis is triggered by the disturbed potential V_p, whereas in our model, we have disturbances of the jump term caused by B^p. This leads to a qualitatively different behaviour of λ_p/p as $p \to \infty$ (see Fig. 2). In particular, there exists a uniform upper bound μ.

The *quenched* Lyapunov exponent for variations of the model with the white noise potential (1.19) has been studied in [2–4] and [9].

1.5. Open problems and extensions of the model

For $p \in \mathbb{N}$, let

$$\kappa_{p,\mathrm{cr}}(\varrho) := \inf\left\{\kappa \geq 0 | \lambda_p(\kappa, \varrho) = 0\right\}$$

denote the critical value for κ above which $\lambda_p(\kappa, \varrho)$ vanishes. It is clear from our results that

$$\kappa_{p,\mathrm{cr}}(\varrho) \nearrow G_d \quad \text{as } p \nearrow \infty,$$

but it is open whether $\kappa_{p,\mathrm{cr}}(\varrho)$ is *strictly* increasing in p for $\varrho > 0$.

Next, one can extend the setting to a multiple catalyst model with a finite number n of catalyst particles. Then the potential ξ has the form

$$\xi(t, x) = \sum_{i=1}^{n} \delta_0\left(x - Y_t^{(i)}\right), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{Z}^d,$$

with $Y^{(1)}, \ldots, Y^{(n)}$ being a collection of n independent random walks with generator $\varrho\Delta$. The degenerate cases $\kappa = 0$ and $\varrho = 0$ can be solved easily, but the general case is more complex than the single catalyst setting. However, the Feynman–Kac formula applied to the solution $u^{(n)}$ of (1.1) with n catalysts yields

$$\langle u^{(n)}(t, 0)\rangle = \mathbb{E}_{0;0,\ldots,0}^{X;Y^1,\ldots,Y^n} \exp\left\{\int_0^t \sum_{i=1}^{n} \delta_0(X_s - Y_{t-s}^i)\,\mathrm{d}s\right\}.$$

Hence the corresponding Lyapunov exponent $\lambda_1^{(n)}$ satisfies the equation

$$\lambda_1^{(n)}(\kappa, \varrho) = \lambda_n^{(1)}(\varrho, \kappa)$$

(cf. (2.5) below). Note that the roles of κ and ϱ are exchanged. Again there exists an operator replacing the role of $\mathcal{H}^p$ in our work, but the study of the upper boundary of its spectrum may turn out to be more complex.

2. Existence and spectral characterization of the Lyapunov exponents

The aim of this section is to prove Theorem 1.2, which links the asymptotic behaviour of $\langle u(t, x)^p\rangle$ as $t \to \infty$ to the ℓ^2-spectrum $\mathrm{Sp}(\mathcal{H}^p)$ of the operator $\mathcal{H}^p$.

Let X_t^i ($i = 1, \ldots, p$) and Y_t be independent random walks on $\mathbb{Z}^d$ with generators $\kappa\Delta$ and $\varrho\Delta$, respectively. Taking notation from (1.6) and (1.7), we note that $\kappa\Delta_1 + \cdots + \kappa\Delta_p + \varrho B^p$ is the generator of a random walk on $\mathbb{Z}^{pd}$ having the form

$$(Z_t^1, \dots, Z_t^p) := (X_t^1 - Y_t, \dots, X_t^p - Y_t). \tag{2.1}$$

Here X_t^i corresponds to a single jump caused by $\kappa \Delta_i$, whereas Y_t corresponds to "diagonal" jumps caused by ϱB^p. Hence we obtain the Feynman–Kac representation of the $\ell^2(\mathbb{Z}^{pd})$-semigroup $\{e^{t\mathcal{H}^p} \mid t \geq 0\}$ generated by $\mathcal{H}^p$ as

$$\left(e^{t\mathcal{H}^p} f\right)(z_1, \dots, z_p) = \mathbb{E}_{z_1,\dots,z_p}^{Z^1,\dots,Z^p} \exp\left\{\int_0^t \sum_{i=1}^p \delta_0(Z_s^i)\, ds\right\} f(Z_t^1, \dots, Z_t^p). \tag{2.2}$$

A natural start for the analysis of $\langle u(t,x)^p\rangle$ is the Feynman–Kac formula (1.3) with $u_0 \equiv 1$. For the potential (1.2) we get

$$u(t,x) = \mathbb{E}_x^X \exp\left\{\int_0^t \delta_{Y_{t-s}}(X_s)\, ds\right\}. \tag{2.3}$$

Using this together with Fubini's theorem we obtain

$$\begin{aligned}\langle u(t,x)^p\rangle &= \mathbb{E}_{x,\dots,x;0}^{X^1,\dots,X^p;Y} \exp\left\{\int_0^t \sum_{i=1}^p \delta_{Y_{t-s}}(X_s^i)\, ds\right\} \\ &= \sum_{z\in\mathbb{Z}^d} \mathbb{E}_{x,\dots,x;0}^{X^1,\dots,X^p;Y} \exp\left\{\int_0^t \sum_{i=1}^p \delta_0(X_s^i - Y_{t-s})\, ds\right\} \delta_z(Y_t),\end{aligned}$$

where $(X_t^1, \dots, X_t^p, Y_t)_{t\geq 0}$ is the joint process of the previously introduced independent random walks $X_t^1, \dots, X_t^p, Y_t$ and $\mathbb{E}_{x_1,\dots,x_p;y}^{X^1,\dots,X^p;Y}$ denotes its expectation when starting at $(x_1, \dots, x_p; y)$. For convenience we use the abbreviation

$$A_t := \int_0^t \sum_{i=1}^p \delta_0(X_s^i - Y_s)\, ds. \tag{2.4}$$

A time reversion of Y yields

$$\langle u(t,x)^p\rangle = \sum_{z\in\mathbb{Z}^d} \mathbb{E}_{x,\dots,x;z}^{X^1,\dots,X^p;Y} \exp\{A_t\}\, \delta_0(Y_t). \tag{2.5}$$

Proceeding from the representation formula (2.5), we prepare the proof of Theorem 1.2. We first show that, although the random field $\xi(t)$ is not spatially shift invariant, the moment Lyapunov exponents are independent of x.

Lemma 2.1. *Let $\kappa + \varrho > 0$, and assume that the limit*

$$\lim_{t\to\infty} \frac{1}{t} \log\langle u(t,0)^p\rangle \tag{2.6}$$

exists. Then, for all $x \in \mathbb{Z}^d$,

$$\lim_{t\to\infty} \frac{1}{t} \log\langle u(t,x)^p\rangle = \lim_{t\to\infty} \frac{1}{t} \log\langle u(t,0)^p\rangle. \tag{2.7}$$

Proof. Fix $y_1, y_2 \in \mathbb{Z}^d$ arbitrarily. We first consider the case $\kappa > 0$. We start with (2.5) and only consider paths $X^1, \ldots, X^p$ that start in y_1 and are at y_2 at time 1 and paths Y that are again at the starting site at time 1. Then we use the Markov property (MP). This yields

$$\begin{aligned}\langle u(t,y_1)^p\rangle \geq\ & \sum_{z\in\mathbb{Z}^d} \mathbb{E}^{X^1,\ldots,X^p;Y}_{y_1,\ldots,y_1;z}\,\delta_{y_2}(X_1^1)\cdots\delta_{y_2}(X_1^p)\,\delta_z(Y_1)\\ &\times \exp\left\{\int_1^t \sum_{i=1}^p \delta_0(X_s^i - Y_s)\,\mathrm{d}s\right\}\delta_0\,(Y_t)\\ \overset{\text{(MP)}}{=}\ & \sum_{z\in\mathbb{Z}^d} \mathbb{P}^{X^1}_{y_1}(X_1^1 = y_2)\cdots\mathbb{P}^{X^p}_{y_1}(X_1^p = y_2)\,\mathbb{P}^Y_z(Y_1 = z)\\ &\times \mathbb{E}^{X^1,\ldots,X^p;Y}_{y_2,\ldots,y_2;z}\exp\left\{\int_0^{t-1}\sum_{i=1}^p \delta_0(X_s^i - Y_s)\,\mathrm{d}s\right\}\delta_0\,(Y_{t-1}).\end{aligned}$$

In the last step, we took into account that $X_t^1, \ldots, X_t^p, Y_t$ are independent. As $X_t^1, \ldots, X_t^p$ are identically distributed and $\mathbb{P}^Y_z(Y_1 = z) \geq \mathrm{e}^{-2d\varrho}$,

$$\langle u(t,y_1)^p\rangle \geq \left[\mathbb{P}^{X^1}_{y_1}(X_1^1 = y_2)\right]^p \mathrm{e}^{-2d\varrho}\langle u(t-1,y_2)^p\rangle.$$

Thus, for $y_1 = x$, $y_2 = 0$,

$$\liminf_{t\to\infty}\frac{1}{t}\log\langle u(t,x)^p\rangle \geq \lim_{t\to\infty}\frac{1}{t}\log\langle u(t,0)^p\rangle,$$

whereas, for $y_1 = 0$, $y_2 = x$,

$$\lim_{t\to\infty}\frac{1}{t}\log\langle u(t,0)^p\rangle \geq \limsup_{t\to\infty}\frac{1}{t}\log\langle u(t,x)^p\rangle.$$

Hence the limit $\lim_{t\to\infty} t^{-1}\log\langle u(t,x)^p\rangle$ exists and coincides with (2.6).

The case $\kappa = 0$ (and hence $\varrho > 0$) follows the same line of arguments. Since $X_s \equiv x$ in the Feynman–Kac representation (2.3),

$$u(t,y_1)^p = \exp\left\{p\int_0^t \delta_0(y_1 - Y_s)\,\mathrm{d}s\right\}.$$

Consequently,

$$\begin{aligned}\mathbb{E}_0^Y u(t,y_1)^p \geq\ & \mathbb{E}_0^Y \exp\left\{p\int_1^t \delta_0(y_1 - Y_s)\,\mathrm{d}s\right\}\delta_0(y_1 - y_2 - Y_1)\\ \overset{\text{(MP)}}{=}\ & \mathbb{P}_0^Y(Y_1 = y_1 - y_2)\,\mathbb{E}^Y_{y_1-y_2}\exp\left\{p\int_0^{t-1}\delta_0(y_1 - Y_s)\,\mathrm{d}s\right\}\\ =\ & \mathbb{P}_0^Y(Y_1 = y_1 - y_2)\,\mathbb{E}^Y_0\exp\left\{p\int_0^{t-1}\delta_0(y_2 - Y_s)\,\mathrm{d}s\right\},\end{aligned}$$

where the last line comes from the spatial shift $y \mapsto y - (y_1 - y_2)$. Therefore,

$$\langle u(t,y_1)^p\rangle \geq \mathbb{P}_0^Y\,(Y_1 = y_1 - y_2)\ \langle u(t-1,y_2)^p\rangle,$$

and, after substituting 0 and x for y_1 and y_2 and taking limits as before, we are done. □

Given $l > 0$, let $Q_l := [-l, l]^d \cap \mathbb{Z}^d$. We need the following lemma to derive the upper bound in the proof of Theorem 1.2. It states that on the right of (2.5) we can restrict to paths that start and end in the finite box $Q_{\ell(t)}$ with

$$\ell(t) := t \log^2 t. \tag{2.8}$$

Lemma 2.2. *As $t \to \infty$,*

$$\langle u(t,0)^p \rangle = (1 + o(1)) \sum_{z \in Q_{\ell(t)}} \mathbb{E}_{0,\dots,0;z}^{X^1,\dots,X^p;Y} \exp\{A_t\}\, \delta_0(Y_t)\, \mathbb{1}_{(X_t^1,\dots,X_t^p) \in Q_{\ell(t)}^p}. \tag{2.9}$$

Proof. It will be sufficient to check that

$$r(t) := \frac{\sum_{z \in \mathbb{Z}^d} \mathbb{E}_{0,\dots,0;z}^{X^1,\dots,X^p;Y} \mathrm{e}^{A_t}\, \delta_0(Y_t) - \sum_{z \in Q_{\ell(t)}} \mathbb{E}_{0,\dots,0;z}^{X^1,\dots,X^p;Y} \mathrm{e}^{A_t}\, \delta_0(Y_t)\, \mathbb{1}_{(X_t^1,\dots,X_t^p) \in Q_{\ell(t)}^p}}{\sum_{z \in \mathbb{Z}^d} \mathbb{E}_{0,\dots,0;z}^{X^1,\dots,X^p;Y} \mathrm{e}^{A_t}\, \delta_0(Y_t)}$$

tends to 0 as $t \to \infty$. Obviously, $r(t) \geq 0$. Splitting the first sum as $\sum_{z \in \mathbb{Z}^d} = \sum_{z \notin Q_{\ell(t)}} + \sum_{z \in Q_{\ell(t)}}$ and then using that $1 \leq \mathrm{e}^{A_t} \leq \mathrm{e}^{pt}$, we obtain

$$\begin{aligned} r(t) &\leq \mathrm{e}^{pt} \frac{\sum_{z \notin Q_{\ell(t)}} \mathbb{E}_{0,\dots,0;z}^{X^1,\dots,X^p;Y} \delta_0(Y_t) + \sum_{z \in Q_{\ell(t)}} \mathbb{E}_{0,\dots,0;z}^{X^1,\dots,X^p;Y} \delta_0(Y_t)\, \mathbb{1}_{(X_t^1,\dots,X_t^p) \notin Q_{\ell(t)}^p}}{\mathbb{E}_{0,\dots,0;0}^{X^1,\dots,X^p;Y} \delta_0(Y_t)} \\ &\leq \mathrm{e}^{pt} \frac{\sum_{z \notin Q_{\ell(t)}} \mathbb{P}_z^Y(Y_t = 0) + \mathbb{P}_{0,\dots,0}^{X^1,\dots,X^p}\left((X_t^1,\dots,X_t^p) \notin Q_{\ell(t)}^p\right)}{\mathbb{P}_0^Y(Y_t = 0)} \\ &= \mathrm{e}^{pt} \frac{\mathbb{P}_0^Y\left(Y_t \notin Q_{\ell(t)}\right) + \mathbb{P}_{0,\dots,0}^{X^1,\dots,X^p}\left((X_t^1,\dots,X_t^p) \notin Q_{\ell(t)}^p\right)}{\mathbb{P}_0^Y(Y_t = 0)}. \end{aligned} \tag{2.10}$$

In the last two transformations we used again a time reversal for Y. For sufficiently large values of t and our choice of $\ell(t)$,

$$\mathbb{P}_0\left(Y_t \notin Q_{\ell(t)}\right) \leq \mathrm{e}^{-\ell(t)}$$

(cf. [8, Lemma 4.3]). The same is true for $X^1, \dots, X^p$ instead of Y. On the other hand, the transition function of a simple random walk decays at most polynomially in time. Hence, on the right hand side of (2.10), the numerator is superexponentially decreasing, but the denominator is (at most) polynomially decreasing. This yields $\lim_{t\to\infty} r(t) = 0$. □

The next lemma is needed to derive the lower bound in the proof of Theorem 1.2. Roughly speaking, it ensures that paths ending outside the finite box $Q_{\ell(t)}$ are asymptotically negligible. It can be seen as a counterpart to Lemma 2.2 with a somewhat modified choice of indicators.

Lemma 2.3. *As $t \to \infty$,*

$$\begin{aligned} &\sum_{y \in Q_{\ell(t)}} \mathbb{E}_{0,\dots,0;0}^{X^1,\dots,X^p;Y} \exp\{A_t\}\, \delta_y(X_t^1) \cdots \delta_y(X_t^p)\, \delta_y(Y_t) \\ &\quad = (1 + o(1)) \sum_{y \in \mathbb{Z}^d} \mathbb{E}_{0,\dots,0;0}^{X^1,\dots,X^p;Y} \exp\{A_t\}\, \delta_y(X_t^1) \cdots \delta_y(X_t^p)\, \delta_y(Y_t). \end{aligned} \tag{2.11}$$

Proof. The proof is similar to that of the previous lemma. We have to show that

$$r(t) := \frac{\sum_{y \notin Q_{\ell(t)}} \mathbb{E}_{0,\dots,0;0}^{X^1,\dots,X^p;Y} \exp\{A_t\}\, \delta_y(X_t^1) \cdots \delta_y(X_t^p)\, \delta_y(Y_t)}{\sum_{y \in \mathbb{Z}^d} \mathbb{E}_{0,\dots,0;0}^{X^1,\dots,X^p;Y} \exp\{A_t\}\, \delta_y(X_t^1) \cdots \delta_y(X_t^p)\, \delta_y(Y_t)}$$

tends to 0 as $t \to \infty$. Again, because of $1 \le \mathrm{e}^{A_t} \le \mathrm{e}^{pt}$, we obtain

$$0 \le r(t) \le \mathrm{e}^{pt}\, \frac{\mathbb{P}_{0,\dots,0}^{X^1,\dots,X^p}\Big((X_t^1, \dots, X_t^p) \notin Q_{\ell(t)}^p\Big)\, \mathbb{P}_0^Y\left(Y_t \notin Q_{\ell(t)}\right)}{\mathbb{P}_{0,\dots,0}^{X^1,\dots,X^p}\left(X_t^1 = 0, \dots, X_t^p = 0\right) \mathbb{P}_0^Y\left(Y_t = 0\right)}.$$

The expression on the right converges to zero as $t \to \infty$ by the same arguments as in the previous proof. □

Now we have collected all ingredients for the proof of Theorem 1.2.

Proof of Theorem 1.2. The proof will be split into two parts:

$$\text{(i)} \quad \limsup_{t\to\infty} \frac{1}{t} \log\langle u(t,0)^p\rangle \le \sup \mathrm{Sp}(\mathcal{H}^p), \tag{2.12}$$

$$\text{(ii)} \quad \liminf_{t\to\infty} \frac{1}{t} \log\langle u(t,0)^p\rangle \ge \sup \mathrm{Sp}(\mathcal{H}^p). \tag{2.13}$$

This together with Lemma 2.1 then proves Theorem 1.2.

(i) *Upper bound*. Since $\mathbb{1}_{(X_t^1,\dots,X_t^p)\in Q_{\ell(t)}^p} \cdot \delta_0(Y_t) \le \mathbb{1}_{(X_t^1 - Y_t,\dots,X_t^p - Y_t)\in Q_{\ell(t)}^p}$, we conclude from Lemma 2.2 that

$$\langle u(t,0)^p\rangle \le (1+o(1)) \sum_{z\in Q_{\ell(t)}} \mathbb{E}_{0,\dots,0;z}^{X^1,\dots,X^p;Y} \exp\{A_t\}\, \mathbb{1}_{(X_t^1 - Y_t,\dots,X_t^p - Y_t)\in Q_{\ell(t)}^p}.$$

Now we apply the transformation (2.1) and the semigroup (2.2) to obtain

$$\begin{aligned}
\langle u(t,0)^p\rangle &\le (1+o(1)) \sum_{z\in Q_{\ell(t)}} \mathbb{E}_{z,\dots,z}^{Z^1,\dots,Z^p} \exp\left\{\int_0^t \sum_{i=1}^p \delta_0(Z_s^i)\,\mathrm{d}s\right\} \mathbb{1}_{(Z_t^1,\dots,Z_t^p)\in Q_{\ell(t)}^p} \\
&\le (1+o(1)) \sum_{z_1,\dots,z_p\in Q_{\ell(t)}} \mathbb{E}_{z_1,\dots,z_p}^{Z^1,\dots,Z^p} \exp\left\{\int_0^t \sum_{i=1}^p \delta_0(Z_s^i)\,\mathrm{d}s\right\} \mathbb{1}_{(Z_t^1,\dots,Z_t^p)\in Q_{\ell(t)}^p} \\
&= (1+o(1)) \left(\mathrm{e}^{t\mathcal{H}^p} \mathbb{1}_{Q_{\ell(t)}^p}, \mathbb{1}_{Q_{\ell(t)}^p}\right),
\end{aligned} \tag{2.14}$$

where $(\cdot,\cdot)$ denotes the inner product in $\ell^2(\mathbb{Z}^{pd})$ with corresponding norm $\|\cdot\|$. Set $\mu := \sup \mathrm{Sp}(\mathcal{H}^p)$ and let $\{E_\lambda; \lambda \le \mu\}$ denote the family of spectral projectors associated with the bounded and self-adjoint operator $\mathcal{H}^p$. Using the spectral representation

$$\mathrm{e}^{t\mathcal{H}^p} = \int_{(-\infty,\mu]} \mathrm{e}^{t\lambda}\,\mathrm{d}E_\lambda,$$

we find that

$$\left(\mathrm{e}^{t\mathcal{H}^p}\mathbb{1}_{Q_{\ell(t)}},\mathbb{1}_{Q_{\ell(t)}}\right)=\int_{(-\infty,\mu]}\mathrm{e}^{t\lambda}\,\mathrm{d}\left(E_\lambda\mathbb{1}_{Q^p_{\ell(t)}},\mathbb{1}_{Q^p_{\ell(t)}}\right)$$
$$\le \mathrm{e}^{t\mu}\int_{(-\infty,\mu]}\mathrm{d}\left\|E_\lambda\mathbb{1}_{Q^p_{\ell(t)}}\right\|^2$$
$$=\mathrm{e}^{t\mu}\,\|\mathbb{1}_{Q^p_{\ell(t)}}\|^2. \tag{2.15}$$

Combining (2.14) and (2.15) we get

$$\langle u(t,0)^p\rangle\le(1+o(1))\;\mathrm{e}^{t\mu}|Q^p_{\ell(t)}|.$$

Since $|Q^p_{\ell(t)}|$ increases only polynomially, this yields the upper bound (2.12).

(ii) *Lower bound*. Restricting the expectation on the right of (2.5) to paths of $X^1,\ldots,X^p,Y$ starting and ending at 0, we get

$$\langle u(t,0)^p\rangle\ge\mathbb{E}^{X^1,\ldots,X^p;Y}_{0,\ldots,0;0}\,\mathrm{e}^{A_t}\,\delta_0(X^1_t)\cdots\delta_0(X^p_t)\,\delta_0(Y_t)$$
$$=\sum_{x_1,\ldots,x_p,y\in\mathbb{Z}^d}\mathbb{E}^{X^1,\ldots,X^p;Y}_{0,\ldots,0;0}\,\mathrm{e}^{A_{t/2}}\,\delta_{x_1}(X^1_{t/2})\cdots\delta_{x_p}(X^p_{t/2})\,\delta_y(Y_{t/2})$$
$$\times\,\mathrm{e}^{A_t-A_{t/2}}\,\delta_0(X^1_t)\cdots\delta_0(X^p_t)\,\delta_0(Y_t). \tag{2.16}$$

An application of the Markov property at time $t/2$ transforms the expression on the right of (2.16) into

$$\sum_{x_1,\ldots,x_p,y\in\mathbb{Z}^d}\mathbb{E}^{X^1,\ldots,X^p;Y}_{0,\ldots,0;0}\,\mathrm{e}^{A_{t/2}}\,\delta_{x_1}(X^1_{t/2})\cdots\delta_{x_p}(X^p_{t/2})\,\delta_y(Y_{t/2})$$
$$\times\,\mathbb{E}^{X^1,\ldots,X^p;Y}_{x_1,\ldots,x_p;y}\,\mathrm{e}^{A_{t/2}}\,\delta_0(X^1_{t/2})\cdots\delta_0(X^p_{t/2})\,\delta_0(Y_{t/2}).$$

After a time reversion in the second line, we may bound this expression from below by

$$\sum_{\substack{x_1,\ldots,x_p,\\ y\in\mathbb{Z}^d}}\left(\mathbb{E}^{X^1,\ldots,X^p;Y}_{0,\ldots,0;0}\,\mathrm{e}^{A_{t/2}}\,\delta_{x_1}(X^1_{t/2})\cdots\delta_{x_p}(X^p_{t/2})\,\delta_y(Y_{t/2})\right)^2$$
$$\ge\sum_{y\in Q_{\ell(t)}}\left(\mathbb{E}^{X^1,\ldots,X^p;Y}_{0,\ldots,0;0}\,\mathrm{e}^{A_{t/2}}\,\delta_y(X^1_{t/2})\cdots\delta_y(X^p_{t/2})\,\delta_y(Y_{t/2})\right)^2.$$

Using the inequality

$$\sum_{i=1}^n x_i^2\ge\frac{1}{n}\left(\sum_{i=1}^n x_i\right)^2,\qquad x_1,\ldots,x_n\in\mathbb{R},$$

and Lemma 2.3, the last expression can further be bounded from below by

$$\frac{1}{|Q_{\ell(t)}|}\left(\sum_{y\in Q_{\ell(t)}}\mathbb{E}^{X^1,\ldots,X^p;Y}_{0,\ldots,0;0}\,\mathrm{e}^{A_{t/2}}\,\delta_y(X^1_{t/2})\cdots\delta_y(X^p_{t/2})\,\delta_y(Y_{t/2})\right)^2$$
$$=\frac{1+o(1)}{|Q_{\ell(t)}|}\left(\sum_{y\in\mathbb{Z}^d}\mathbb{E}^{X^1,\ldots,X^p;Y}_{0,\ldots,0;0}\,\mathrm{e}^{A_{t/2}}\,\delta_y(X^1_{t/2})\cdots\delta_y(X^p_{t/2})\,\delta_y(Y_{t/2})\right)^2,$$

$$= \frac{1+o(1)}{|Q_{\ell(t)}|} \left(\mathbb{E}_{0,\ldots,0;0}^{X^1,\ldots,X^p;Y} \mathrm{e}^{A_{t/2}} \delta_0(X^1_{t/2} - Y_{t/2}) \cdots \delta_0(X^p_{t/2} - Y_{t/2}) \right)^2 .$$

As before, applying the transformation (2.1) and collecting the above bounds, we arrive at

$$\langle u(t,0)^p \rangle \geq \frac{1+o(1)}{|Q_{\ell(t)}|} \left(\mathbb{E}_{0,\ldots,0}^{Z^1,\ldots,Z^p} \exp\left\{ \int_0^{t/2} \sum_{i=1}^p \delta_0(Z^i_s)\, \mathrm{d}s \right\} \delta_0(Z^1_{t/2}) \cdots \delta_0(Z^p_{t/2}) \right)^2 . \tag{2.17}$$

Again, expressing (2.17) with the help of the semigroup (2.2), we obtain

$$\langle u(t,0)^p \rangle \geq \frac{1+o(1)}{|Q_{\ell(t)}|} \left(\mathrm{e}^{(t/2)\mathcal{H}^p} \delta_0, \delta_0 \right)^2 . \tag{2.18}$$

In order to find a lower bound for the expression on the right of (2.18), we restrict the ℓ^2-operator $\mathcal{H}^p$ to a finite box with Dirichlet boundary condition and apply the Perron–Frobenius theory for nonnegative irreducible matrices. This is done as follows.

By killing the process $(Z^1_t, \ldots, Z^p_t)$ upon leaving the box $Q^p_n = [-n,n]^{pd} \cap \mathbb{Z}^{pd}$, we get a new semigroup in $\ell^2(Q^p_n)$ with generator $\mathcal{H}^p_n$ acting on $f \in \ell^2(Q^p_n)$ as

$$\left(\mathrm{e}^{t\mathcal{H}^p_n} f \right)(z_1,\ldots,z_p) = \mathbb{E}_{z_1,\ldots,z_p}^{Z^1,\ldots,Z^p} \exp\left\{ \int_0^t \sum_{i=1}^p \delta_0(Z^i_s)\, \mathrm{d}s \right\} f(Z^1_t,\ldots,Z^p_t)\, \mathbb{1}_{\tau_Q > t}, \tag{2.19}$$

where $(z_1,\ldots,z_p) \in Q^p_n$ and $\tau_Q := \inf\{t | (Z^1_t,\ldots,Z^p_t) \notin Q^p_n\}$ denotes the first exit time from the box Q^p_n. Accordingly, for all $f \in \ell^2(Q^p_n)$,

$$\mathcal{H}^p_n f(z_1,\ldots,z_p) = \mathcal{H}^p \widehat{f}_n(z_1,\ldots,z_p), \quad (z_1,\ldots,z_p) \in Q^p_n, \tag{2.20}$$

where

$$\widehat{f}_n = \begin{cases} f & \text{on } Q^p_n, \\ 0 & \text{on } \mathbb{Z}^{pd} \setminus Q^p_n. \end{cases}$$

Furthermore, for any $\varepsilon > 0$, $\mathcal{H}^p_n + (2d\kappa + \varepsilon)\, I$ is a positive operator that obeys the prerequisites of the Perron–Frobenius theorem, where I is the identical operator. Hence there exists a strictly positive eigenfunction v_n with $\|v_n\| = 1$, corresponding to the largest eigenvalue of $\mathcal{H}^p_n + (2d\kappa + \varepsilon)\, I$ having multiplicity 1. Then v_n is also an eigenfunction to the largest eigenvalue μ_n of $\mathcal{H}^p_n$ and an eigenfunction to the largest eigenvalue $\mathrm{e}^{(t/2)\,\mu_n}$ of $\mathrm{e}^{(t/2)\,\mathcal{H}^p_n}$ having multiplicity 1. Denote by $\{E^n_\lambda; \lambda \leq \mu_n\}$ the family of spectral projectors associated with the operator $\mathcal{H}^p_n$. Using again the spectral representation, we obtain

$$\begin{aligned} \left(\mathrm{e}^{(t/2)\mathcal{H}^p_n} \delta_0, \delta_0 \right) &= \mathrm{e}^{(t/2)\mu_n} (v_n, \delta_0)^2 + \int_{(-\infty,\mu_n)} \mathrm{e}^{(t/2)\lambda}\, \mathrm{d}\left(E^n_\lambda \delta_0, \delta_0 \right) \\ &\geq \mathrm{e}^{(t/2)\mu_n} v_n(0)^2 . \end{aligned}$$

Since $v_n(0)$ is positive, the above inequality implies that

$$\liminf_{t\to\infty} \frac{1}{t} \log \left(\mathrm{e}^{(t/2)\mathcal{H}^p_n} \delta_0, \delta_0 \right) \geq \frac{\mu_n}{2}. \tag{2.21}$$

We combine the inequalities (2.18) and (2.21) with the semigroups (2.2) and (2.19) to obtain for all $n \in \mathbb{N}$ that

$$\begin{aligned}\liminf_{t\to\infty} \frac{1}{t} \log \left\langle u(t,0)^p \right\rangle &\geq \liminf_{t\to\infty} \frac{1}{t} \log \left\{ \frac{1+o(1)}{|Q_{\ell(t)}|} \left(\mathrm{e}^{(t/2)\mathcal{H}^p} \delta_0, \delta_0 \right)^2 \right\} \\ &= 2 \liminf_{t\to\infty} \frac{1}{t} \log \left(\mathrm{e}^{(t/2)\mathcal{H}^p} \delta_0, \delta_0 \right) \\ &\geq 2 \liminf_{t\to\infty} \frac{1}{t} \log \left(\mathrm{e}^{(t/2)\mathcal{H}_n^p} \delta_0, \delta_0 \right) \\ &\geq \mu_n.\end{aligned}$$

It remains to show that

$$\lim_{n\to\infty} \mu_n = \mu.$$

By the Rayleigh–Ritz formula for μ_n and (2.20),

$$\mu_n = \sup_{f \in \ell^2(Q_n^p),\, \|f\|=1} \left(\mathcal{H}_n^p f, f \right) = \sup_{\substack{f \in \ell^2(\mathbb{Z}^{pd}),\, \|f\|=1 \\ \mathrm{supp}(f) \subset Q_n^p}} \left(\mathcal{H}^p f, f \right). \tag{2.22}$$

Here $\mathrm{supp}(f)$ denotes the support of f. We see from (2.22) that μ_n is nondecreasing in n. Let $f \in \ell^2(\mathbb{Z}^{pd})$. Then $f \mathbb{1}_{Q_n^p} \to f$ in the norm sense and, since $\mathcal{H}^p$ is a bounded linear operator, $(\mathcal{H}^p(f\mathbb{1}_{Q_n^p}), f\mathbb{1}_{Q_n^p}) \to (\mathcal{H}^p f, f)$. This validates

$$\sup_{\|f\|=1} \left(\mathcal{H}^p f, f \right) = \sup_{\substack{\|f\|=1 \\ |\mathrm{supp}(f)|<\infty}} \left(\mathcal{H}^p f, f \right).$$

Together with (2.22), we obtain the desired equality

$$\begin{aligned}\mu &= \sup_{\|f\|=1} \left(\mathcal{H}^p f, f \right) = \sup_{n\in\mathbb{N}} \sup_{\substack{\|f\|=1 \\ \mathrm{supp}(f) \subset Q_n^p}} \left(\mathcal{H}^p f, f \right) \\ &= \sup_{n\in\mathbb{N}} \mu_n = \lim_{n\to\infty} \mu_n.\end{aligned}$$

This completes the proof. □

3. Analysis of the Lyapunov exponents and intermittency

In this section we study the behaviour of λ_p for varying $p \in \mathbb{N}$ under the influence of the system parameters κ and ϱ and analyse the intermittency behaviour of the system to prove Theorems 1.5 and 1.6. In Section 3.1 we prove some standard statements that hold quite generally for any (nonnegative) version of the potential ξ. In Section 3.2 we prove some preliminary results for the degenerate cases $\varrho = 0$ and $\kappa = 0$, being of crucial importance for Section 3.3, where we prove Theorem 1.5. Finally, Section 3.4 is devoted to the proof of Theorem 1.6.

3.1. General relations between Lyapunov exponents

In this section we study the general situation where we assume that ξ is any nonnegative potential and that the Lyapunov exponents (1.4) exist for all $p \in \mathbb{N}$.

Lemma 3.1 (*General Properties of Lyapunov Exponents*).

(i) *For all* $p \in \mathbb{N}$,

$$\frac{\lambda_p}{p} \le \frac{\lambda_{p+1}}{p+1};$$

(ii) *the mapping* $p \mapsto \lambda_p$ *is convex, i.e., for all* $p, q \in \mathbb{N}$ *and* $\alpha \in (0, 1)$ *with* $\alpha p+(1-\alpha)q \in \mathbb{N}$,

$$\lambda_{\alpha p+(1-\alpha)q} \le \alpha\lambda_p + (1-\alpha)\lambda_q;$$

(iii) *if* $\lambda_p/p < \lambda_{p+1}/(p+1)$ *for some* $p \in \mathbb{N}$, *then* $\lambda_q/q < \lambda_{q+1}/(q+1)$ *for all* $q \in \mathbb{N}$ *with* $q > p$.

Proof. (i) The first assertion is obvious from the moment inequality

$$\langle u(t,x)^p\rangle^{\frac{1}{p}} \le \langle u(t,x)^{p+1}\rangle^{\frac{1}{p+1}}$$

and the definition (1.4) of the Lyapunov exponents.

(ii) Let $\alpha \in (0, 1)$ and $p, q, \alpha p + (1-\alpha)q \in \mathbb{N}$. By Hölder's inequality,

$$\langle u(t,x)^{\alpha p+(1-\alpha)q}\rangle \le \langle u(t,x)^p\rangle^{\alpha}\ \langle u(t,x)^q\rangle^{1-\alpha}.$$

This implies the desired inequality.

(iii) It is sufficient to show the assertion for $q = p + 1$. We proceed indirectly by assuming that $\lambda_p/p < \lambda_{p+1}/(p+1)$ but $\lambda_{p+1}/(p+1) = \lambda_{p+2}/(p+2)$. Then, by assertion (ii),

$$\lambda_{p+1} \le \frac{1}{2}\lambda_p + \frac{1}{2}\lambda_{p+2} < \frac{1}{2}\left(\frac{p}{p+1}\lambda_{p+1} + \frac{p+2}{p+1}\lambda_{p+1}\right) = \lambda_{p+1},$$

which is a contradiction. □

Remark. We had to restrict the convexity to those $\alpha \in (0, 1)$ with $\alpha p + (1-\alpha)q \in \mathbb{N}$, because we only know the existence of λ_p for $p \in \mathbb{N}$.

3.2. The degenerate cases $\kappa = 0$ *and* $\varrho = 0$

We now return to the case where the random potential ξ has the form (1.2). We will first prove Lemma 1.3 treating the degenerate case $\varrho = 0$.

Proof of Lemma 1.3. If $\varrho = 0$, then $\xi(t,x) = \delta_0(x)$ and the solution u to (1.1) is deterministic. Hence $\lambda_p(\kappa, 0) = p\lambda_1(\kappa, 0)$, and the assertion of the lemma follows with (1.13). □

We now consider the case $\kappa = 0$.

Proof of Lemma 1.4. We use the Feynman–Kac formula (1.3) with $X_s \equiv 0$ and a simple time scaling to see that $\lambda_p(0,\varrho) = p\lambda_1(0,\varrho/p)$. Combining this with (1.13), we arrive at the desired assertion. □

3.3. Properties of the Lyapunov exponents $\lambda_p(\kappa,\varrho)$

In this subsection we will prove Theorem 1.5.

Proof of Theorem 1.5. (i) Fix $p \in \mathbb{N}$. With the help of Theorem 1.2 and the Rayleigh–Ritz formula we can write

$$\begin{aligned}\lambda_p(\kappa,\varrho) &= \sup \operatorname{Sp}\left(\mathcal{H}^p\right) = \sup_{\substack{f\in\ell^2(\mathbb{Z}^{pd})\\ \|f\|=1}} \left(\mathcal{H}^p f, f\right)\\ &= \sup_{\substack{f\in\ell^2(\mathbb{Z}^{pd})\\ \|f\|=1}} \Big[\kappa\left((\Delta_1+\cdots+\Delta_p)f,f\right)+\varrho\left(B^p f,f\right)+\Big((\delta_0^{(1)}+\cdots+\delta_0^{(p)})f,f\Big)\Big].\end{aligned} \tag{3.1}$$

Hence, as a supremum of linear functions of κ and ϱ, $\lambda_p(\kappa,\varrho)$ is convex and lower semicontinuous. Since every finite convex function on $[0,\infty)^2$ is upper semicontinuous, we get the desired continuity. Monotonicity follows directly, because the first two inner products in (3.1) are nonpositive. It remains to show that λ_p vanishes if $\kappa \geq G_d$. By monotonicity and Lemma 1.3, $0 \leq \lambda_p(\kappa,\varrho) \leq \lambda_p(\kappa,0) = p\,\mu(\kappa)$, but the right hand side equals 0 if $\kappa \geq G_d$, by (1.11).

(ii) Fix $\kappa,\varrho \geq 0$ arbitrarily. By Lemma 3.1, $\lambda_p(\kappa,\varrho)/p$ is nondecreasing in p. As in (i), Theorem 1.2 and the Rayleigh–Ritz formula yield

$$\lambda_p(\kappa,\varrho) = \sup_{\substack{f\in\ell^2(\mathbb{Z}^{pd})\\ \|f\|=1}} \Big[\Big((\kappa\Delta_1+\cdots+\kappa\Delta_p+\delta_0^{(1)}+\cdots+\delta_0^{(p)})f,f\Big)+\left(\varrho B^p f,f\right)\Big]. \tag{3.2}$$

On the other hand, by Lemma 1.3,

$$p\,\mu(\kappa) = \lambda_p(\kappa,0) = \sup_{\substack{f\in\ell^2(\mathbb{Z}^{pd})\\ \|f\|=1}} \Big((\kappa\Delta_1+\cdots+\kappa\Delta_p+\delta_0^{(1)}+\cdots+\delta_0^{(p)})f,f\Big). \tag{3.3}$$

From (3.2) and (3.3) we conclude that

$$\left|\frac{\lambda_p(\kappa,\varrho)}{p}-\mu(\kappa)\right| \leq \frac{\varrho}{p} \sup_{\substack{f\in\ell^2(\mathbb{Z}^{pd})\\ \|f\|=1}} |(B^p f,f)|. \tag{3.4}$$

Hence, to prove the convergence (1.18), it suffices to show that the supremum on the right stays bounded as $p \to \infty$. We write e_i for the i-th unit vector in $\mathbb{Z}^d$. For arbitrary $f \in \ell^2(\mathbb{Z}^{pd})$, we obtain

$$\begin{aligned}B^p f(x_1,\ldots,x_p) &= \sum_{i=1}^d \left[f(x_1+e_i,\ldots,x_p+e_i)-f(x_1,\ldots,x_p)\right]\\ &\quad+\sum_{i=1}^d \left[f(x_1-e_i,\ldots,x_p-e_i)-f(x_1,\ldots,x_p)\right].\end{aligned}$$

Using a spatial shift in the second line we can compute the Dirichlet form associated with the operator B^p:

$$-\left(B^p f,f\right) = \sum_{i=1}^d \sum_{x_1,\ldots,x_p\in\mathbb{Z}^d} \left[f(x_1+e_i,\ldots,x_p+e_i)-f(x_1,\ldots,x_p)\right]^2. \tag{3.5}$$

In particular, $(B^p f, f) \le 0$. Using the inequality $(a-b)^2 \le 2a^2 + 2b^2$ we conclude from (3.5) that

$$\sup_{\|f\|=1} |(B^p f, f)| \le 4d,$$

and we are done. □

3.4. Intermittency

Finally, we want to analyse the intermittency behaviour of the system by proving Theorem 1.6. To this end, we need the following lemma.

Lemma 3.2. *If $\varrho > 0$ and $\kappa + \varrho < G_d$, then $\lambda_2/2 > \lambda_1$, i.e., the system shows full intermittency.*

Proof. Since $\lambda_1(\kappa, \varrho) = \mu(\kappa + \varrho)$ and $\kappa + \varrho < G_d$, λ_1 is positive and the largest eigenvalue of the operator $\mathcal{H}^1 = (\kappa + \varrho)\Delta + \delta_0$ corresponding to a positive eigenfunction v with $\|v\| = 1$. Then $(v \otimes v)(x, y) = v(x)\, v(y)$ is an eigenfunction of the operator

$$\widetilde{\mathcal{H}^2} = \mathcal{H}^1 \otimes \mathcal{H}^1 = (\kappa + \varrho)(\Delta_1 + \Delta_2) + (\delta_0^{(1)} + \delta_0^{(2)}),$$

corresponding to the eigenvalue $2\lambda_1$. Using the Rayleigh–Ritz formula, we conclude that

$$\begin{aligned}\lambda_2 - 2\lambda_1 &= \sup \mathrm{Sp}(\mathcal{H}^2) - \sup \mathrm{Sp}(\widetilde{\mathcal{H}^2}) \\ &= \sup_{\|f\|=1} \left(\mathcal{H}^2 f, f\right) - \left(\widetilde{\mathcal{H}^2}\, v \otimes v, v \otimes v\right) \\ &\ge \left(\left(\mathcal{H}^2 - \widetilde{\mathcal{H}^2}\right) v \otimes v, v \otimes v\right).\end{aligned}$$

But

$$\begin{aligned}&\left(\left(\mathcal{H}^2 - \widetilde{\mathcal{H}^2}\right) v \otimes v, v \otimes v\right) \\ &= \varrho \left(\left(B^2 - \Delta_1 - \Delta_2\right) v \otimes v, v \otimes v\right) \\ &= 2\varrho \sum_{x,y \in \mathbb{Z}^d} \sum_{i=1}^d [v(x)v(y+e_i) - v(x)v(y)]\, [v(x-e_i)v(y) - v(x)v(y)] \\ &= 2\varrho \sum_{x,y \in \mathbb{Z}^d} \sum_{i=1}^d v(x)\, [v(x-e_i) - v(x)]\, v(y)\, [v(y+e_i) - v(y)] \\ &= \varrho \sum_{x,y \in \mathbb{Z}^d} \sum_{i=1}^d [v(x-e_i) - v(x)]^2\, [v(y+e_i) - v(y)]^2 \\ &= \varrho \sum_{i=1}^d \left(\sum_{x \in \mathbb{Z}^d} [v(x-e_i) - v(x)]^2\right)^2.\end{aligned}$$

Assume that the above expression vanishes. Then v is constant. Since $v \in \ell^2(\mathbb{Z}^d)$, this implies $v \equiv 0$, which contradicts $\|v\| = 1$. Therefore, $\lambda_2 - 2\lambda_1 > 0$. □

Proof of Theorem 1.6. Let $\varrho > 0$. We first consider the case $\kappa + \varrho < G_d$. Then $\lambda_2/2 > \lambda_1$ by Lemma 3.2. Hence, $\lambda_{p+1}/(p+1) > \lambda_p/p$ for all $p \in \mathbb{N}$ by Lemma 3.1, and the system is fully intermittent.

Next, consider the case $G_d - \varrho \leq \kappa < G_d$. By Theorem 1.2 and (1.11), we see that in this case $\lambda_1(\kappa, \varrho) = \mu(\kappa + \varrho) = 0$, whereas $\mu(\kappa) > 0$. Theorem 1.5 yields the convergence $\lambda_p(\kappa, \varrho)/p \nearrow \mu(\kappa)$ as $p \to \infty$. Hence, there exists $p \in \mathbb{N}$ such that $\lambda_p(\kappa, \varrho) > 0$. Set $p^* := \min\{p \in \mathbb{N} | \lambda_p(\kappa, \varrho) > 0\}$. Then the system is p^*-intermittent.

There remains the case $\kappa \geq G_d$. Then $\lambda_1(\kappa, \varrho) = \lambda_2(\kappa, \varrho) = \cdots = 0$ by Theorem 1.5(i), and the system is not intermittent. □

Acknowledgement

The second author was partially supported by the Netherlands Organisation for Scientific Research (NWO).

References

[1] R.A. Carmona, S.A. Molchanov, Parabolic Anderson Model and Intermittency, Memoirs of the American Mathematical Society (Nr 518) (1994).

[2] R.A. Carmona, S.A. Molchanov, Stationary parabolic Anderson model and intermittency, Probab. Theory Related Fields 102 (1995) 433–453.

[3] R.A. Carmona, S.A. Molchanov, F. Viens, Sharp upper bound on the almost-sure exponential behavior of a stochastic partial differential equation, Random Oper. Stochastic Equations 4 (1996) 43–49.

[4] M. Cranston, T.S. Mountford, T. Shiga, Lyapunov exponents for the parabolic Anderson model, Acta Math. Univ. Comenian. LXXI (2002) 163–188.

[5] W. Ebeling, A. Engel, B. Esser, R. Feistel, Diffusion and reaction in random media and models of evolution processes, J. Stat. Phys. 37 (1984) 369–385.

[6] J. Gärtner, F. den Hollander, Intermittency in a catalytic random medium. EURANDOM Report 2004-019, 2004. Ann. Probab. (in press).

[7] J. Gärtner, W. König, The parabolic Anderson model, in: J.-D. Deuschel, A. Greven (Eds.), Interacting Stochastic Systems, Springer, 2005, pp. 153–179.

[8] J. Gärtner, S.A. Molchanov, Parabolic problems for the Anderson model, Commun. Math. Phys. 132 (1990) 613–655.

[9] H. Kesten, V. Sidoravicius, Branching random walk with catalysts, Electron. J. Probab. 8 (Paper no. 5) (2003) 1–51.

[10] Ya.B. Zel'dovich, S.A. Molchanov, S.A. Ruzmajkin, D.D. Sokolov, Intermittency, diffusion and generation in a nonstationary random medium, Sov. Sci. Rev. Sect. C, Math. Phys. Rev. 7 (1988) 1–110.